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so many that, the number being squared and the product quadrupled, the result will be 90,000." Required to know how many goats he had.[1]

A DETERMINATION OF THE CURVE MINIMIZING THE AREA ENCLOSED BY IT AND ITS EVOLUTE.

By OTTO DUNKEL, Washington University, St. Louis, Mo.

One of the problems treated in the calculus of variations as an example in which the second derivative appears in the integrand is that of determining the curve in a plane passing through two fixed points which with its evolute and its two normals of given directions at these two points enclose a minimum area.[2] The special form of the integrand permits this problem to be solved without resorting to the general theory of the calculus of variations, and the conditions appear as necessary and sufficient simultaneously. The solution given here appears to be new so far as the writer can learn from the references consulted. It is adapted only to this special form of the integrand, but it is quite possible that there may be other problems of this form to which it applies. For example, the problem of determining the curve which with its caustic produced by parallel rays of light, and the reflected rays at two of its given points, enclose a minimum area may also be solved in the same way.[3] It will be observed that the method admits of a slight generalization.

The Necessary and Sufficient Conditions for a Minimum. Let one of the fixed points be the origin and the other (x_2, y_2) and let the inclination of the curve to the y-axis at these two points be θ_1 and θ_2, respectively, $\theta_2 > \theta_1$. Let s be the length of the arc of the curve measured from the origin and, R, the radius of curvature. Then

$$R = \frac{ds}{d\theta} \tag{1}$$

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[1] In the above brief extracts the archaic forms of expression have been retained so far as the circumstances of translation permit. No effort has been made to explain, in this presentation, the method of attacking the problems, or to consider the sources from which the author drew his materials.

[2] This problem is discussed in many well-known works, for example: I. Todhunter, *Researches in the Calculus of Variations*, London, 1871, chapter 13; H. Hancock, *Lectures on the Calculus of Variations*, Cincinnati, 1904, pp. 75–76; A. Kneser, *Lehrbuch der Variationsrechnung*, Braunschweig, 1900, pp. 203, 219; and O. Bolza, *Vorlesungen über Variationsrechnung*, Leipzig, 1909, p. 152. It originated with Euler (*Methodus Inveniendi Lineas Curvas* . . . Lausannæ & Genevæ, 1744, pp. 64–66): "Invenire curvam *Am*, quæ cum sua evoluta *AR* & radio osculi *mR* in quavis loco applicato, minimum spatium *ARm* includat"; the solution which Euler gives employs the calculus of variations. In *Annals of Mathematics*, new series, vol. 14, pp. 14–26, 1912, E. J. Miles discusses the "Determination of the constants in Euler's problem concerning the minimum area between a curve and its evolute," and in this MONTHLY, *1917*, 420–422, P. R. Rider discusses "An intrinsic equation solution of a problem of Euler."

[3] A paper on this problem was read by Professor Dunkel before the American Mathematical Society, November 27, 1920.—EDITOR..

determines R as a positive quantity, as we shall suppose that the curve is always concave downward. We shall also assume that R is a continuous function of θ. The integral for the area, S, between the curve, its evolute, and the two normals at the end points

$$S = \frac{1}{2}\int_{\theta_1}^{\theta_2} R^2 d\theta \tag{2}$$

is to be made a minimum subject to the conditions

$$x_2 = \int_{\theta_1}^{\theta_2} R \sin\theta\, d\theta, \qquad y_2 = \int_{\theta_1}^{\theta_2} R\cos\theta\, d\theta. \tag{3}$$

The problem is handled more easily by replacing these by equivalent conditions involving similar integrals equated to zero. This is easily accomplished by setting

$$R = A\sin\theta + B\cos\theta + \eta, \tag{4}$$

where A and B are constants to be determined so that

$$\begin{aligned} x_2 &= A\int_{\theta_1}^{\theta_2} \sin^2\theta\, d\theta + B\int_{\theta_1}^{\theta_2} \sin\theta\cos\theta\, d\theta,\\ y_2 &= A\int_{\theta_1}^{\theta_2} \sin\theta\cos\theta\, d\theta + B\int_{\theta_1}^{\theta_2} \cos^2\theta\, d\theta,\\ 0 &= \int_{\theta_1}^{\theta_2} \eta\sin\theta\, d\theta, \qquad 0 = \int_{\theta_1}^{\theta_2} \eta\cos\theta\, d\theta. \end{aligned} \tag{4'}$$

The last two of these four equations follow from the first two and (3) and (4). The first pair of equations can always be solved for A and B, since their determinant is never zero. This may be seen as follows: the integral

$$\int_{\theta_1}^{\theta_2} (\lambda\sin\theta + \cos\theta)^2\, d\theta$$

is a quadratic form in λ which is always positive; also the coefficient of λ^2 is always positive. Hence its discriminant is positive; but this discriminant is the determinant of the pair of equations in (4′).[1] The equation (2) is now replaced by

$$S = \frac{1}{2}\int_{\theta_1}^{\theta_2} (A\sin\theta + B\cos\theta)^2\, d\theta + \frac{1}{2}\int_{\theta_1}^{\theta_2} \eta^2\, d\theta, \tag{2'}$$

where two integrals are omitted since they are each zero by virtue of the new conditions on η in the last line of (4′). Since R is continuous, η is continuous, and it is immediately obvious that $\eta = 0$ is a necessary and a sufficient condition for a minimum. This value of η satisfies the two equations in (4′). Hence the curve for which

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[1] It may be shown by computation that the value of the determinant is $[(\theta_2 - \theta_1)^2 - \sin^2(\theta_2 - \theta_1)]/4 > 0$, but the above method is given in preference on account of its generality. It should be noticed that the method does not depend upon the special functions $\sin\theta$ and $\cos\theta$.

$$R = A \sin \theta + B \cos \theta, \tag{5}$$

where A and B are given by (4′), gives a minimum area.

The Minimizing Curve and its Evolute. The equations for x and y differ from those in the first two lines of (4′) by having θ, x, y in place of θ_2, x_2, y_2. Since not both x_2 and y_2 are zero, A and B cannot both be zero and hence we may set $A = 4a \cos \alpha$, $B = 4a \sin \alpha$, $\psi = \alpha + \theta$, where $4a = \sqrt{A^2 + B^2}$. Inserting these in (5) and rotating the axes through the angle α, we find $R = 4a \sin \psi$, where ψ is the angle that the tangent to the curve makes with the new y'-axis. The integration of the new differential equations,

$$dx' = 4a \sin^2 \psi d\psi, \qquad dy' = 4a \sin \psi \cos \psi d\psi,$$

gives

$$\begin{aligned} x' + a[2(\theta_1 + \alpha) - \sin 2(\theta_1 + \alpha)] &= a(2\psi - \sin 2\psi), \\ y' + a[1 - \cos 2(\theta_1 + \alpha)] &= a(1 - \cos 2\psi). \end{aligned} \tag{6}$$

Hence the curve is a cycloid generated by a circle of radius a rolling on a straight line parallel to the x'-axis and at a distance $a[1 - \cos 2(\theta_1 + \alpha)]$ below it with a cusp at $x' = - a[2(\theta_1 + \alpha) - \sin 2(\theta_1 + \alpha)]$ and at intervals of $2\pi a$. The angle through which the circle has rolled is given by 2ψ.

If σ is the length of arc of the evolute, the radius of curvature of the evolute is

$$\rho = \frac{d\sigma}{d\psi} = 4a \cos \psi = 4a \sin \left(\frac{\pi}{2} + \psi \right).$$

The values of ψ which make R vanish determine the cusps of the cycloid and hence these cusps must be points of the evolute. Moreover at such points of the evolute ρ has its maximum absolute value $4a$. This shows that the evolute is an equal cycloid having the same direction and with its vertices at the cusps of the former.[1]

Fixed End Points and a Variable Slope at One End. We now consider the case in which the end points are fixed, the slope at the initial point is given but that at the final point is not assigned. If there is a curve C which gives a minimum area, a necessary condition may be obtained as follows. Suppose that the inclination of C at the terminal point is θ_2 and let us consider only those curves $[C]$ which have this terminal inclination and satisfy the remaining conditions. Since the area given by C does not exceed that resulting from any one of the curves $[C]$, the previous work shows that the curve C must be a cycloid. It remains now to determine the terminal inclination θ_2 of a cycloid which satisfies all the other end conditions and which gives a smaller area than any cycloid having a different terminal inclination. This is merely a problem of minimizing a definite function S of θ_2. Returning to the equations (2), (4′), (5) the expression for the area becomes

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[1] This result was given by Huygens in *Christiani Hvgenii . . . Horologivm Oscillatorivm . . .*, Paris, 1673, page 67, prop. VI: "Semicycloidis evolutione, à vertice cœpta, alia semicyclois describitur evolutæ æqualis & similis, cujus basis est in ea recta quæ cycloidem evolutam in vertice contingit."—EDITOR.

$$\begin{aligned}(7)\qquad 2S &= A^2\int_{\theta_1}^{\theta_2}\sin^2\theta\,d\theta + 2AB\int_{\theta_1}^{\theta_2}\sin\theta\cos\theta\,d\theta + B^2\int_{\theta_1}^{\theta_2}\cos^2\theta\,d\theta,\\ &= Ax_2 + By_2,\end{aligned}$$

where A and B are functions of θ_2. Hence

$$(8)\qquad 2\frac{dS}{d\theta_2} = x_2\frac{dA}{d\theta_2} + y_2\frac{dB}{d\theta_2}.$$

From (4′) we have

$$(9)\qquad \begin{aligned}0 &= \frac{dA}{d\theta_2}\int_{\theta_1}^{\theta_2}\sin^2\theta\,d\theta + \frac{dB}{d\theta_2}\int_{\theta_1}^{\theta_2}\sin\theta\cos\theta\,d\theta + A\sin^2\theta_2 + B\sin\theta_2\cos\theta_2,\\ 0 &= \frac{dA}{d\theta_2}\int_{\theta_1}^{\theta_2}\sin\theta\cos\theta\,d\theta + \frac{dB}{d\theta_2}\int_{\theta_1}^{\theta_2}\cos^2\theta\,d\theta + A\sin\theta_2\cos\theta_2 + B\cos^2\theta_2.\end{aligned}$$

By multiplying the first of these equations by A, the second by B, and adding the results, we have, after using (4′) and (5)

$$x_2\frac{dA}{d\theta_2} + y_2\frac{dB}{d\theta_2} + R_2^2 = 0,$$

where R_2 is the radius of curvature of the cycloid at (x_2, y_2). Hence

$$(10)\qquad 2\frac{dS}{d\theta_2} = -R_2^2,$$

and therefore the area decreases as θ_2 increases. There is therefore no finite value of θ_2 rendering S a minimum. We may say, however, that for a portion of a single arch the cycloid having a cusp at the terminal point gives a smaller area than any cycloid having a smaller terminal inclination. From this it follows that the portion of a single arch of a cycloid having a terminal cusp gives a smaller area than any other curve having the same or a smaller inclination at the terminal point.

If we suppose now that the inclination at the terminal point is given, but that at the initial point is variable, it will be found that

$$(10')\qquad 2\frac{dS}{d\theta_1} = R_1^2,$$

where R_1 is the radius of curvature of the cycloid at the initial point. Hence the area decreases as θ_1 decreases.

Consider now the case in which the inclination at each end point is variable. The area S decreases as θ_1 decreases and θ_2 increases. Suppose for simplicity that $y_2 = 0$; then a cycloid with the respective inclinations 0 and π at the end points, *i.e.*, a single arch with cusps at the end points, gives a smaller area than any portion of an arch with intermediate inclinations at these same end points. It thus follows that the area given by a single complete arch of a cycloid is smaller than that given by any other curve whose initial inclination is not smaller than 0 and whose terminal inclination is not greater than π.

The method employed above applies to integrals of the form (2) with any number of conditions such as (3) in which other functions of θ as well as the trigonometric may appear in the integrand. Certain light restrictions are to be placed upon such functions.

QUESTIONS AND DISCUSSIONS.

Edited by W. A. Hurwitz, Cornell University, Ithaca, N. Y.

REPLIES.

34 [*1917*, 134, 341; *1920*, 114, 301, 405, 460]. Given the mixed integral and functional equation

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

I. Remarks by J. P. Ballantine, Pennsylvania State College.

In remarks by the editor on this question [*1920*, 302], attention was called to the related equation

$$\int_{\alpha}^{\beta} f(x)dx = \frac{\beta - \alpha}{6}\left[f(\alpha) + 4f\left(\frac{\alpha + \beta}{2}\right) + f(\beta)\right], \tag{1}$$

where both α and β are allowed to vary; and it was pointed out that if $f(x)$ is a solution of this equation possessing a fifth derivative, then $f(x)$ must be a polynomial of degree $\leqq 3$. It will be proved here that the same conclusion follows from the assumption that $f(x)$ is merely continuous.

I. *Let x_0, x_2, x_4, x_6 be four equally spaced points, and x_1, x_3, x_5 the points of bisection of the three equal intervals formed. Then any integrable function which satisfies* (1) *and vanishes at x_0, x_2, x_4, x_6 vanishes also at x_1, x_3, x_5.*

Let $x_2 - x_0 = 2a$. We find from (1)

$$\int_{x_0}^{x_4} f(x)dx = 0, \qquad \int_{x_2}^{x_6} f(x)dx = 0.$$

Hence, with the notation

$$\int_{x_2}^{x_4} f(x)dx = G, \tag{2}$$

we find

$$\int_{x_0}^{x_2} f(x)dx = -G, \tag{3}$$

$$\int_{x_4}^{x_6} f(x)dx = -G; \tag{4}$$

and therefore, by addition,

$$\int_{x_0}^{x_6} f(x)dx = -G + G - G = -G. \tag{5}$$

By (1) and (2),

$$G = \tfrac{4}{3}af(x_3);$$